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The Euclidean path integrals and the Feynman–Dyson–Wick perturbation expansion

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Abstract. We derive the Feynman rules for the complex ϕ^4 theory using the Euclidean path integral formulation.

Many years ago Feynman (1948) derived his rules for quantum electrodynamics (QED) using heuristic arguments. The derivations were later refined by Dyson (1949) and Wick (1950), but their methods were very complicated. I would like to show, in this short paper, how easily the Feynman rules can follow from the path integral formulation.

I have in mind a complex ϕ^4 theory with Lagrangian $\hat{\mathscr{L}} = \partial_\mu\hat{\phi}^\dagger\partial^\mu\hat{\phi} - \mu^2\hat{\phi}^\dagger\phi - g(\hat{\phi}^\dagger\hat{\phi})^2/4!$, and I choose to look at the perturbation expansion of

$$\tau(x_1, x_2, x_3, x_4) \equiv \langle 0|T[\hat{\phi}^\dagger(x_1)\hat{\phi}^\dagger(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)]|0\rangle$$

($\langle 0|$ here means the ground state of the theory) as an illustration.

In order to represent $\tau(x_1, x_2, x_3, x_4)$ by path integrals, I first look at the matrix element

$$\langle\sigma'(\boldsymbol{x}), T'|T[\hat{\phi}^\dagger(x_1)\hat{\phi}^\dagger(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)]|\sigma(\boldsymbol{x}), T\rangle$$

which can be represented as (Feynman and Hibbs 1965):

$$\int_{\substack{\phi(T',\boldsymbol{x})=\sigma'(\boldsymbol{x})\\ \phi(T,\boldsymbol{x})=\sigma(\boldsymbol{x})}} \mathscr{D}[\phi]\phi^*(x_1)\phi^*(x_2)\phi(x_3)\phi(x_4)\exp\left(\mathrm{i}\int_T^{T'}\mathrm{d}t\int \mathrm{d}\boldsymbol{x}\mathscr{L}\right).$$

Inserting complete sets of energy eigenstates, rotating time into an imaginary axis ($t = -\mathrm{i}\lambda$, $T' = -\mathrm{i}\Lambda'$, $T = -\mathrm{i}\Lambda$), and finally letting $\Lambda' \to \infty$, $\Lambda \to -\infty$ isolates the ground state contribution†:

$$\begin{aligned}\tau^{\mathrm{E}}(x_1, x_2, x_3, x_4)\Psi_0^{\mathrm{E}}[\sigma'(\boldsymbol{x})]\Psi_0^{*\mathrm{E}}[\sigma(\boldsymbol{x})]\exp[-E_0(\Lambda'-\Lambda)]\\ = \int_{\substack{\phi(-\infty,\boldsymbol{x})=\sigma'(\boldsymbol{x})\\ \phi(+\infty,\boldsymbol{x})=\sigma(\boldsymbol{x})}} \mathscr{D}[\phi]\phi^*(x_1)\phi^*(x_2)\phi(x_3)\phi(x_4)\exp\left(\int_{-\infty}^{+\infty}\mathrm{d}^4x\mathscr{L}^{\mathrm{E}}\right)\end{aligned} \tag{1}$$

where E signifies the space to be Euclidean, and Ψ_0 is the ground state functional.

† The rotation into the imaginary axis in order to project out the ground state expectation values was discussed in the report by Abers and Lee (1973). Our derivation of the Feynman rules is more illuminating and the results are already in the momentum space.

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Using the same tricks, we can show that

$$\Psi_0^{\mathrm{E}}[\sigma'(\boldsymbol{x})]\Psi_0^{*\mathrm{E}}[\sigma(\boldsymbol{x})]\,\mathrm{e}^{-E_0(\Lambda'-\Lambda)}=\int_{\substack{\phi(-\infty,\boldsymbol{x})=\sigma'(\boldsymbol{x})\\ \phi(+\infty,\boldsymbol{x})=\sigma(\boldsymbol{x})}}\mathscr{D}[\phi]\exp\biggl(\int_{-\infty}^{+\infty}\mathrm{d}^4x\mathscr{L}^{\mathrm{E}}\biggr). \tag{2}$$

Hence, by combining equations (1) and (2), we obtain the Euclidean path integral representation of $\tau^{\mathrm{E}}(x_1, x_2, x_3, x_4)$:

$$\begin{aligned}\tau^{\mathrm{E}}&(x_1, x_2, x_3, x_4)\\ &=\biggl[\int_{\substack{\phi(-\infty,\boldsymbol{x})=\sigma'(\boldsymbol{x})\\ \phi(+\infty,\boldsymbol{x})=\sigma(\boldsymbol{x})}}\mathscr{D}[\phi]\phi^*(x_1)\phi^*(x_2)\phi(x_3)\phi(x_4)\exp\biggl(\int_{-\infty}^{+\infty}\mathrm{d}^4x\mathscr{L}^{\mathrm{E}}\biggr)\biggr]\\ &\quad\times\biggl[\int_{\substack{\phi(-\infty,\boldsymbol{x})=\sigma'(\boldsymbol{x})\\ \phi(+\infty,\boldsymbol{x})=\sigma(\boldsymbol{x})}}\mathscr{D}[\phi]\exp\biggl(\int_{-\infty}^{\infty}\mathrm{d}^4x\mathscr{L}^{\mathrm{E}}\biggr)\biggr]^{-1}.\end{aligned} \tag{3}$$

The perturbation expansion of τ^{E} comes from the term by term integration of the path integrals:

$$\begin{aligned}\tau^{\mathrm{E}}&(x_1, x_2, x_3, x_4)_{\mathrm{pert}}\\ &=\biggl\{\int_{\substack{\phi(-\infty,\boldsymbol{x})=\sigma'(\boldsymbol{x})\\ \phi(+\infty,\boldsymbol{x})=\sigma(\boldsymbol{x})}}\mathscr{D}[\phi]\phi^*(x_1)\phi^*(x_2)\phi(x_3)\phi(x_4)\\ &\quad\times\sum_{n=0}^{\infty}\frac{1}{n!}\biggl(\frac{-g}{4!}\int(\phi^*\phi)^2\,\mathrm{d}^4x\biggr)^n\exp\biggl[-\int\phi^*\biggl(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\nabla^2+\mu^2\biggr)\phi\,\mathrm{d}^4x\biggr]\biggr\}\\ &\quad\times\biggl\{\int_{\substack{\phi(-\infty,\boldsymbol{x})=\sigma'(\boldsymbol{x})\\ \phi(+\infty,\boldsymbol{x})=\sigma(\boldsymbol{x})}}\mathscr{D}[\phi]\sum_{n=0}^{\infty}\frac{1}{n!}\biggl(\frac{-g}{4!}\int(\phi^*\phi)^2\,\mathrm{d}^4x\biggr)^n\\ &\quad\times\exp\biggl[-\int\phi^*\biggl(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\nabla^2+\mu^2\biggr)\phi\,\mathrm{d}^4x\biggr]\biggr\}^{-1}.\end{aligned} \tag{4}$$

Each term in equation (4) is Gaussian, and the path integration may be done by first solving the following eigenvalue problem:

$$\begin{aligned}&\biggl(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\nabla^2+\mu^2\biggr)\phi_n(x)=\lambda_n^2\phi_n(x)\\ &\phi_n(-\infty,\boldsymbol{x})=\sigma'(\boldsymbol{x})\\ &\phi_n(+\infty,\boldsymbol{x})=\sigma(\boldsymbol{x})\end{aligned} \tag{5}$$

and expanding $\phi(x)$ in terms of $\phi_n(x)$,

$$\phi(x)=\sum_{n=0}^{\infty}\xi_n\phi_n(x). \tag{6}$$

The measure $\mathscr{D}[\phi]$ will then be $\Pi_{n=0}^{\infty}\,\mathrm{d}\xi_n$.

For the time being forget about the boundary conditions; I shall come back to this issue later. The obvious solutions for equation (5) are the plane waves

e^{-ikx} with $\lambda_k^2 = k^2+\mu^2 = \boldsymbol{k}^2+k_0^2+\mu^2$, and hence $\phi(x)$ may be expanded to give

$$\phi(x)=\int \frac{d^4k}{(2\pi)^4}\xi_k\, e^{-ikx}. \tag{7}$$

Equation (4) can then be written as

$$\begin{aligned}
&\tau^{E}(x_1, x_2, x_3, x_4)_{\text{pert}}\\
&\quad=\left\{\int_0^\infty [\Pi\, d\xi]\xi_n^*\xi_m^*\xi_p\xi_g\, e^{i(k_nx_1+k_mx_2-k_px_3-k_qx_4)}\right.\\
&\qquad\left.\times\left[\sum_{l=0}^\infty \frac{1}{l!}\left(-\frac{g}{4!}\xi_\alpha^*\xi_\beta\xi_\gamma^*\xi_\rho\delta^4(k_\alpha-k_\beta+k_\gamma-k_\rho)\right)^l\right] e^{-\lambda_s^2\xi_s^*\xi_s}\right\}\\
&\qquad\times\left\{\int_0^\infty [\Pi\, d\xi]\left[\sum_{l=0}^\infty \frac{1}{l!}\left(-\frac{g}{4!}\xi_\alpha^*\xi_\beta\xi_\gamma^*\xi_\rho\delta^4(k_\alpha-k_\beta+k_\gamma-k_\rho)\right)^l\right] e^{-\lambda_s^2\xi_s^*\xi_s}\right\}^{-1}.
\end{aligned} \tag{8}$$

In the above equation integrations are replaced by summations for notational simplicity and repeated indices are used to denote summation.

Everything will become clear if we evaluate the first-order contribution to $\tau^{E}(x_1, x_2, x_3, x_4)_{\text{pert}}$ according to (8),

$$\begin{aligned}
&\tau^{E}(x_1, x_2, x_3, x_4)^{(1)}_{\text{pert}}\\
&\quad=\left(-\frac{g}{4!}\int_0^\infty [\Pi\, d\xi]\xi_n^*\xi_m^*\xi_p\xi_q\, e^{i(k_nx_1+k_mx_2-k_px_3-k_qx_4)}\xi_\alpha^*\xi_\beta\xi_\gamma^*\xi_\rho\right.\\
&\qquad\left.\times\,\delta^4(k_\alpha-k_\beta+k_\gamma-k_\rho)\, e^{-\lambda_s^2\xi_s^*\xi_s}\right)\left(\int_0^\infty [\Pi\, d\xi]\, e^{-\lambda_s^2\xi_s^*\xi_s}\right)^{-1}.
\end{aligned} \tag{9}$$

Because ξ is complex, it may be written as $\eta\, e^{i\theta}$ and the measure $\Pi\, d\xi$ can be chosen as $\Pi\, \eta\, d\eta\, d\theta$. Then

$$\begin{aligned}
&\tau^{E}(x_1, x_2, x_3, x_4)^{(1)}_{\text{pert}}\\
&\quad=-\frac{g}{4!}\left(\int_0^\infty [\Pi\eta\, d\eta\, d\theta\, e^{-\lambda^2\eta^2}]\eta_n\eta_m\eta_p\eta_q\eta_\alpha\eta_\beta\eta_\gamma\eta_\rho\right.\\
&\qquad\left.\times\, e^{-i(\theta_n+\theta_m-\theta_p-\theta_q+\theta_\alpha-\theta_\beta+\theta_\gamma-\theta_\rho)}\, e^{i(k_nx_1+k_mx_2-k_px_3-k_qx_4)}\delta^4(k_\alpha-k_\beta+k_\gamma-k_\rho)\right)\\
&\qquad\times\left(\int_0^\infty [\Pi\eta\, d\eta\, d\theta\, e^{-\lambda^2\eta^2}]\right)^{-1}.
\end{aligned} \tag{10}$$

The crucial observation comes from the integration of the angular variable θ: the integral will be zero unless $\theta_n+\theta_m-\theta_p-\theta_q+\theta_\alpha-\theta_\beta+\theta_\gamma-\theta_\rho=0$. In other words, the non-vanishing contribution comes from those terms whose θ's are 'paired off'. Looking more carefully at the signs before the k's in

$$\delta^4(k_\alpha-k_\beta+k_\gamma-k_\rho)\, e^{i(k_nx_1+k_mx_2-k_px_3-k_qx_4)},$$

and remembering that $+\theta$ comes from ξ and $-\theta$ comes from ξ^*, it can be seen immediately that the 'pairing off' of the θ is equivalent to the contraction between the creation and annihilation operators used by Wick in his derivation of the Feynman rules.

Corresponding to the employment of the normal ordered Lagrangian by Dyson and Wick, it is postulated that there will be no 'pairing off' of the θ between θ_α, θ_β, θ_γ and θ_ρ which come from the Lagrangian.

The integration is then carried out using the result that

$$\sum_{\substack{\text{perm}\\ \alpha\beta\gamma\rho}} \int_0^\infty [\Pi\eta \,\mathrm{d}\eta \,\mathrm{d}\theta \,\mathrm{e}^{-\lambda^2\eta^2}]\eta_\alpha^2\eta_\beta^2\eta_\gamma^2\eta_\rho^2 = 4!\frac{1}{\lambda_\alpha^2}\frac{1}{\lambda_\beta^2}\frac{1}{\lambda_\gamma^2}\frac{1}{\lambda_\rho^2}\prod_{n=0}^{\infty}\frac{2\pi}{2\sqrt{\lambda_n^2}}. \tag{11}$$

The result in equation (11) is independent of whether some of the α, β, γ and ρ are equal.

The factor $\Pi_{n=0}^{\infty} 2\pi/2\sqrt{\lambda_n^2}$ is cancelled between the numerator and denominator in equation (10), and, since $1/\lambda_k^2 = 1/(\boldsymbol{k}^2+\mu^2+\omega^2)$ is just the Euclidean propagator, the final result is

$$\begin{aligned}\tau^{\mathrm{E}}(x_1, x_2, x_3, x_4)^{(1)}_{\mathrm{pert}} &= -\mathrm{i}g\int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{d}^4k_3\,\mathrm{d}^4k_4}{(2\pi)^8}\frac{1}{k_1^2+\mu^2}\frac{1}{k_2^2+\mu^2}\frac{1}{k_3^2+\mu^2}\frac{1}{k_4^2+\mu^2}\\ &\quad\times \mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2-k_3x_3-k_4x_4)}.\end{aligned} \tag{12}$$

This is exactly what is found if the Feynman–Dyson–Wick scheme is used (of course, a rotation back into the Minkowski region must be made before comparing with physics).

The higher-order contributions will come out if the same arguments as before are used alongside the integral

$$\int_0^\infty x^{2n+1}\,\mathrm{e}^{-px^2}\,\mathrm{d}x = \frac{n!}{2p^{n+1}}. \tag{13}$$

Thus I conclude that equation (8) is just the Gell-Mann–Low formula and that the perturbation expansion is simply the Feynman–Dyson–Wick expansion.

As a final remark, consider the boundary conditions imposed in equation (5). I may choose $\sigma'(\boldsymbol{x}) = \sigma(\boldsymbol{x}) = 0$, and attach a factor $\mathrm{e}^{-\epsilon' t}$ to each e^{-ikx} (ϵ' is infinitesimal and $\epsilon' > 0$ for $t > 0$ and $\epsilon' < 0$ for $t < 0$). In that case, I shall satisfy the right boundary conditions in sacrificing the orthogonality condition. A heuristic way to avoid this problem is to attach $\mathrm{e}^{-\epsilon' t}$ to e^{-ikx} only in the determination of λ_k^2. Then

$$\lambda_k^2 = \boldsymbol{k}^2+\mu^2+(\omega-\mathrm{i}\epsilon')^2 = \boldsymbol{k}^2+\mu^2+\omega^2-\mathrm{i}\epsilon \tag{14}$$

This is always the propagator used.

References

Abers E S and Lee B W 1973 *Phys. Rep.* **90** 1

Dyson F J 1949 *Phys. Rev.* **75** 486

Feynman R P 1948 *Phys. Rev.* **74** 939, 1430

Feynman R P and Hibbs 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)

Wick G C 1950 *Phys. Rev.* **80** 268